

Convolution spline approximations for time domain boundary integral equations

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Abstract

We introduce new “convolution spline” methods for the temporal approximation of time domain boundary integral equations (TDBIEs). They share some properties of convolution quadrature, but instead of being based on an underlying ODE solver are explicitly constructed in terms of basis functions which have compact support, and hence give sparse system matrices. They are derived and analysed for convolution Volterra integral equations (VIEs): at time step $t_n = nh$ the VIE solution is approximated in a “backward time” manner in terms of basis functions ϕ_j by $u(t_n - t) \approx \sum_{j=0}^n u_{n-j} \phi_j(t/h)$ for $t \in [0, t_n]$. We show that using isogeometric B-splines of degree $m \geq 1$ on $[0, \infty)$ in this framework gives a second order accurate scheme, but cubic splines with the “parabolic runout” conditions at $t = 0$ are fourth order accurate. We establish a methodology for the stability analysis of VIEs and demonstrate that the new methods are stable for non-smooth kernels which are related to convergence analysis for TDBIEs, including the case of a Bessel function kernel oscillating at frequency $\mathcal{O}(1/h)$. Numerical results for VIEs and for TDBIE problems on both open and closed surfaces confirm the theoretical predictions.

Keywords: Convolution quadrature, Volterra integral equations, time dependent boundary integral equations

AMS(MOS) subject classification: 65R20, 65M12

1 Introduction

We derive and test new “convolution spline” time-stepping methods for the time-dependent boundary integral equation (TDBIE)

$$\frac{1}{4\pi} \int_{\Gamma} \frac{u(\mathbf{x}', t - |\mathbf{x}' - \mathbf{x}|)}{|\mathbf{x}' - \mathbf{x}|} d\mathbf{x}' = a(\mathbf{x}, t) \quad \text{for } \mathbf{x} \in \Gamma, t > 0 \quad (1.1)$$

for u . It is the single layer potential equation for acoustic scattering from the surface $\Gamma \subset \mathbb{R}^3$ with zero Dirichlet boundary conditions and (known) incident field $-a(\mathbf{x}, t)$, and is equivalent to

$$\int_0^t \int_{\Gamma} k(\mathbf{x}' - \mathbf{x}, t - t') u(\mathbf{x}', t') d\mathbf{x}' dt' = a(\mathbf{x}, t) \quad \text{for } k(\mathbf{z}, t) = \frac{\delta(t - |\mathbf{z}|)}{4\pi |\mathbf{z}|}.$$

Two key properties of the new methods are: (i) they are easy to implement, and (ii) they are computationally efficient (because the resulting system matrices are sparse). They share some properties with convolution quadrature (CQ), and we follow Lubich (1988) in deriving them as approximation schemes for the convolution-kernel Volterra integral equation (VIE)

$$\int_0^t K(t') u(t - t') dt' = a(t), \quad t \in [0, T]. \quad (1.2)$$

Convergence and stability properties of the new schemes are analysed for VIEs, but the focus of the paper is not on deriving new methods for VIEs (of which there are already very many), but on using the insight gained from VIEs to derive new methods which have good properties for TDBIEs.

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1.1 Properties of TDBIE approximations

Designing a good approximation scheme for the TDBIE (1.1) is nontrivial; challenges include ensuring that it is numerically stable, it is not prohibitively hard to implement for a given scattering surface Γ , and its computational complexity is not infeasibly high. There are good survey articles on approximation methods for TDBIEs by Ha-Duong (2003) and Costabel (2004), and we briefly summarise the pros and cons of some of the main approaches.

Bamberger and Ha Duong (1986) proved that a full Galerkin approximation of (1.1) in time and space is stable and convergent for smooth, closed Γ (this was extended to the case of open, flat Γ in Ha-Duong (1990)), but the stability of the method relies on all the integrals being evaluated very accurately (the key insight on how to do this was provided by Terrasse (1993)). In practice this involves converting five dimensional volume integrals over irregular (non-polygonal) sub-regions of $\Gamma \times \Gamma \times [0, T]$ to surface integrals which are then evaluated using high precision quadrature, and is extremely complicated to successfully implement in practice, even for relatively simple Γ . Collocation schemes for (1.1) are far more straightforward to implement, but there is little rigorous convergence analysis for them, and numerical instability is often an issue. Methods which use a Galerkin approximation in space and CQ in time have obvious attractions: they are based on rigorous theoretical analysis (Lubich 1994) and are relatively straightforward to implement. They are also inherently far more stable than those which use Galerkin or collocation time approximations (Lubich (1994) showed that the CQ method remains stable when the inner product integrals are approximated), but unfortunately the disadvantage this time is higher computational complexity.

All three approaches approximate (1.1) as a convolution sum of the form $\sum_{m=0}^n \mathbf{Q}^m \underline{U}^{n-m} = \underline{a}^n$, which is rearranged to give the time-stepping scheme

$$\mathbf{Q}^0 \underline{U}^n = \underline{a}^n - \sum_{m=1}^n \mathbf{Q}^m \underline{U}^{n-m} \quad (1.3)$$

for $\underline{U}^n \in \mathbb{R}^{N_S}$, the representation of the spatial approximation of u at or near time $t^n = nh$, where the right-hand side vector \underline{a}^n is derived from $a(\mathbf{x}, t)$. In the case of both Galerkin and collocation approximations the matrices $\mathbf{Q}^m \in \mathbb{R}^{N_S \times N_S}$ are sparse – the number of nonzero elements per row of matrix \mathbf{Q}^m is $\mathcal{O}(\min\{m, N_S^{1/2}\})$. In particular this means that (1.3) can be solved in $\mathcal{O}(N_S^{3/2})$ operations once the right-hand side is known, and the overall computational complexity to obtain the approximate solution up to time $N_T h$ is $\mathcal{O}(\min\{N_T^3 N_S, N_T^2 N_S^{3/2}\})$ operations. For these hyperbolic problems it is usual to use a timestep h commensurate with the side Δx of a typical space mesh element, and in this case $N_T \approx N$ and $N_S \approx N^2$ for $N = 1/\Delta x$, and the total computational complexity is $\mathcal{O}(N^5)$. Although this compares somewhat unfavourably with the $\mathcal{O}(N^4)$ computational complexity of a finite difference or finite element approximation of the PDE formulation of the acoustic wave equation in \mathbb{R}^3 , the plane wave “fast” methods developed by Michielssen and co-workers (Ergin, Shanker & Michielssen 1999, Ergin, Shanker & Michielssen 2000, Lu, Wang, Ergin & Michielssen 2000) reduces the complexity to $\mathcal{O}(N^3 \log^2 N)$.

Using CQ in time results in a solution algorithm (1.3) in which the matrices \mathbf{Q}^m are dense, because the underlying basis functions are global (see e.g. Sec. 2 below, Hackbusch, Kress & Sauter (2009), or Banjai (2010) for more details), which increases the computational complexity to $\mathcal{O}(N_S^2 N_T^2)$. The issue is not solving (1.3) for \underline{U}^n (which can typically be done efficiently by approximating \mathbf{Q}^0 appropriately), but in performing the matrix–vector products needed to calculate the right-hand side. Lubich explains that the technique of Hairer, Lubich & Schlichte (1985) can be used to reduce the overall complexity to $\mathcal{O}(N_S^2 N_T \log^2 N_T)$, i.e. $\mathcal{O}(N^5 \log^2 N)$. A systematic cut-off strategy to replace small matrix entries by zero is described and carefully analysed in Hackbusch et al. (2009), and this reduces the storage costs of the method. This is combined with panel clustering by Kress & Sauter (2008) to further reduce the storage costs. CQ methods which are based on underlying Runge–Kutta ODE solvers have been developed and analysed for TDBIEs (Banjai & Lubich 2011, Banjai, Lubich & Melenk 2011). There are several advantage of these methods over linear multistep CQ methods: the basis functions are more highly concentrated (Banjai 2010, Figs 1–2), which makes sparsifying the \mathbf{Q}^m matrices more straightforward; and higher order accurate methods in time are possible. Banjai (2010) uses this approach to develop a practical, parallelizable solution algorithm for (1.1) which he illustrates with a number of realistic large-scale numerical examples.

1.2 New convolution spline methods

We describe how to construct a new “convolution spline” method for timestepping (1.1) which shares some properties of CQ, but instead of being based on an underlying ODE solver is explicitly constructed in terms

of basis functions which have compact support. This means that the \mathbf{Q}^m system matrices in (1.3) for our new method have the same sparsity pattern as for the Galerkin or collocation approximations described above, giving a TDBIE solution scheme whose overall complexity is $\mathcal{O}(\min\{N_T^3 N_S, N_T^2 N_S^{3/2}\}) = \mathcal{O}(N^5)$ operations (and which could also be potentially speeded up using “fast” methods). It is also far easier to implement than the full space–time Galerkin approach.

We derive the new approximation as a solution method for the VIE (1.2), with u approximated in terms of B-spline basis functions in a “backwards time” framework. Our initial approach is to use isogeometric B-splines of degree m on $[0, \infty)$. There can be advantages in using higher order values of m even though the formal convergence rate of this scheme for a smooth VIE problem is limited to second order (because it is based on quasi-interpolation by the Schoenberg B-spline operator). For example, as noted in Sauter & Veit (2011a), using smooth temporal basis functions greatly simplifies approximating the integrals in (1.1). We also consider cubic B-splines with the “parabolic runout” condition at $t = 0$ and show that these are fourth order accurate. We carefully test out the new methods on (1.2), establishing formal convergence, and examining the behaviour for kernels which mimic some of the important properties of TDBIE problems, such as discontinuous step-function kernels (see e.g. Sauter & Veit (2011b)). Another important test problem is obtained from taking the spatial Fourier transform of (1.1) at frequency $\omega \in \mathbb{R}^2$ when $\Gamma = \mathbb{R}^2$. This is

$$\int_0^t J_0(\omega t') \hat{u}(\omega, t - t') dt' = 2\hat{u}(\omega, t), \quad (1.4)$$

where $\omega = |\omega|$ and J_0 is the first kind Bessel function of order zero. As noted in Davies (1994), instabilities of approximation schemes for (1.1) are typically exhibited at the highest spatial frequency which can be represented on the mesh. Hence it is important to ensure that any prototype numerical scheme for time-stepping (1.1) is stable for (1.4) at values of $\omega = \mathcal{O}(1/h)$ (assuming $h \approx \Delta x$).

1.3 Outline

Section 2 contains an alternative derivation of Lubich’s (1988) CQ method for (1.2) in terms of basis functions which have the sum to unity property (2.12). The new “convolution spline” approximation of (1.2) is described in Section 3 in terms of basis functions which have compact support and are (essentially) all translates, and we give sufficient conditions for this approximation to be stable. We consider the case in which the basis functions are m th degree isogeometric B-splines on $[0, \infty)$ in Section 4, showing how Laplace transform techniques can be used to prove the stability of this approximation of (1.2) for several different test kernels, and demonstrating second order convergence for (1.2) when K and a satisfy

$$a \in C^{d+1}[0, T], \quad K \in C^{d+1}[0, T], \quad a(0) = 0 \text{ and } K(0) = 1 \quad (1.5)$$

for suitable $d \geq 0$. Under these assumptions, equation (1.2) possesses a unique solution $u \in C^d[0, T]$ – e.g. see, Brunner (2004, Theorem 2.1.9).

In Section 5 we consider a cubic convolution spline basis which is modified near $t = 0$ to satisfy the “parabolic runout” conditions, and show that this gives a far more stable approximation of (1.2) which is fourth order convergent. Numerical tests indicate that this new scheme performs far better than CQ based on BDF2, exhibiting fourth order accuracy even for a discontinuous kernel.

We present numerical test results for TDBIEs in Section 6 which use a Galerkin approximation in space (based on triangular piecewise constant elements), and the cubic convolution spline basis in time, for both open and closed surfaces Γ . The TDBIE test problems are similar to those considered in Davies & Duncan (2013) which use the convolution-in-time framework with non-polynomial (global) basis functions, but the modified B-spline basis functions give a more accurate temporal approximation.

2 CQ based on linear multistep methods for (1.2)

We begin by outlining the derivation of the CQ method for (1.2) from Lubich (1988), in order to show how it can be reinterpreted in terms of CQ basis functions. For simplicity we restrict attention to the case for which the extension of the solution u by zero to the negative real axis is in $C^d(-\infty, T]$ (otherwise the CQ method needs to be ‘corrected’ as described in Lubich (1988, Sec. 3) in order to attain optimal convergence). This is guaranteed by requiring

$$a^{(p)}(0) = 0 \quad \text{for } p = 0 : d + 1 \quad (2.1)$$

because $u^{(p)}(0) = a^{(p+1)}(0) - \sum_{\ell=0}^{p-1} K^{(p-\ell)}(0) u^{(\ell)}(0)$. We also assume that the Laplace transform $\overline{K}(s)$ of the kernel K is sufficiently well-behaved for all the formal manipulations in the next subsection to be rigorous. For details see for example (Lubich 1994, Sec. 1) or (Banjai 2010, App).

2.1 Lubich's CQ method

We follow Lubich (1988) and substitute the Laplace inversion formula for $\overline{K}(s)$ into (1.2) to obtain

$$a(t) = \frac{1}{2\pi i} \int_{\gamma} \overline{K}(s) y(t, s) ds, \quad (2.2)$$

where γ is an infinite contour within the region of analyticity of $\overline{K}(s)$ and $y(t, s) = \int_0^t e^{st'} u(t-t') dt'$. Treating the Laplace variable s as a parameter, $y(t)$ solves the ODE:

$$\dot{y}(t) = s y(t) + u(t), \quad y(0) = 0, \quad (2.3)$$

and this is approximated by the k -step ($k \leq d$) linear multistep method with timestep h

$$\sum_{j=0}^k \alpha_j y_{n+j-k} = h \sum_{j=0}^k \beta_j f_{n+j-k}, \quad (2.4)$$

where $t_n = nh$, $y_n \approx y(t_n)$ and $f_n = s y_n + u(t_n)$. The starting values are $y_{-k} = \dots y_{-1} = 0$ because of the assumption (2.1). Multiplying (2.4) by ξ^n and summing over n (for $\xi \in \mathbb{C}$ for which the sum converges) gives

$$\left(\frac{\delta(\xi)}{h} - s \right) \sum_{n=0}^{\infty} y_n \xi^n = \sum_{n=0}^{\infty} u(t_n) \xi^n, \quad \text{where} \quad \delta(\xi) = \sum_{j=0}^k \alpha_j \xi^{k-j} / \sum_{j=0}^k \beta_j \xi^{k-j}$$

is the symbol of (2.4). Hence y_n is the coefficient of ξ^n in the expansion of $\left(\frac{\delta(\xi)}{h} - s \right)^{-1} \sum_{k=0}^{\infty} u(t_k) \xi^k$. Substituting y_n for $y(t_n)$ in (2.2) shows that $a(t_n)$ is approximated by the coefficient of ξ^n in

$$\frac{1}{2\pi i} \int_{\gamma} \left(\frac{\delta(\xi)}{h} - s \right)^{-1} \overline{K}(s) ds \sum_{k=0}^{\infty} u(t_k) \xi^k = \overline{K}(\delta(\xi)/h) \sum_{k=0}^{\infty} u(t_k) \xi^k$$

using Cauchy's integral formula. Hence, defining the CQ weights $q_k = q_k(h)$ to be the coefficients in the expansion

$$\overline{K}(\delta(\xi)/h) = \sum_{k=0}^{\infty} q_k \xi^k \quad (2.5)$$

gives the CQ approximation of (1.2)

$$a(t_n) = \sum_{j=0}^n q_j u_{n-j}. \quad (2.6)$$

This can be rearranged to give the time-stepping approximate solution $u_n \approx u(t_n)$

$$u_n = \frac{1}{q_0} \left(a(t_n) - \sum_{j=1}^{n-1} q_j u_{n-j} \right) \quad \text{for } n \geq 1, \quad (2.7)$$

since by assumption $u_0 = u(0) = 0$.

2.2 Derivation of CQ in terms of basis functions

The CQ approximation scheme (2.6) for the VIE (1.2) is defined solely in terms of the weights q_k . But if CQ is used to time-step a TDBIE, then the approximation involves ‘‘CQ basis functions’’ – see e.g. (Hackbusch et al. 2009, Banjai 2010, Monegato, Scuderi & Stanic 2011). However, we are not aware of a general interpretation of CQ approximation schemes for (1.2) in terms of basis functions. As well as yielding some interesting observations, this also gives the framework which we use for the derivation of our ‘‘convolution spline’’ methods in Sections 4–5

At $t = t_n := nh$ (1.2) can be written as

$$a(t_n) = \int_0^\infty K(t') u(t_n - t') dt', \quad (2.8)$$

because $u(t) = 0$ for $t \leq 0$. We show below that the standard CQ method is equivalent to approximating u in (2.8) by

$$u(t_n - t') \approx \sum_{j=0}^n u_{n-j} \phi_j(t'/h) \quad \text{for } t' \geq 0 \quad (2.9)$$

where ϕ_j are basis functions. This is **fundamentally different** from standard finite-element type approximations which have the form $u(t) \approx \sum_{k=0}^N u_k \psi_k(t/h)$ – i.e. the basis function ψ_k is always associated with the same “unknown” u_k . (Although note that when all the ψ_k are translates, as happens for some low order schemes, then the two approaches can be similar; see Section 3 for more details.)

Substituting (2.9) into (2.8) and comparing the resulting expression with (2.6) gives the relationship between the standard CQ weights and basis functions:

$$q_j = \int_0^\infty K(t) \phi_j(t/h) dt. \quad (2.10)$$

Comparing this with the standard CQ definition of q_j in (2.5) gives (see Banjai (2010, Eq. (3.1)))

$$e^{-\delta(\xi)t} = \sum_{j=0}^\infty \phi_j(t) \xi^j. \quad (2.11)$$

An immediate consequence is that the basis functions satisfy the sum to unity property

$$\sum_{j=0}^\infty \phi_j(t) = 1, \quad (2.12)$$

provided the underlying multistep ODE solver is consistent, because in this case $\delta(1) = 0$. This new observation is a crucial property which we use in Section 3.

2.3 CQ basis functions for LMMs

Explicit formulae for the $\phi_j(t)$ based on BDF1–2 have been used for TDBIE approximations (Hackbusch et al. 2009, Monegato et al. 2011). The formula for $m = 1$ is given in Monegato et al. (2011), and in this case $\phi_j(t) = e^{-t} t^j / j!$, i.e. they are Erlang functions, used in statistics as probability density functions and satisfy $\phi_j(t) \geq 0$ and $\int_0^\infty \phi_j(t) dt = 1$. The derivation for BDF2 is more complicated, and the explicit formula

$$\phi_j(t) = \frac{1}{j!} H_j(\sqrt{2t}) \left(\frac{t}{2}\right)^{j/2} e^{-3t/2}$$

is given in Hackbusch et al. (2009), where H_j is the j th Hermite polynomial. Note that the properties of H_j imply that $\phi_j(t)$ involves a j th degree polynomial and an exponential in t with no fractional powers of t .

In principle (2.11) can be used directly to find the basis functions $\phi_j(t)$ corresponding to any underlying linear multistep ODE method for (2.3), although this may not be easy in practice. For the trapezoidal rule $\delta(\xi) = 2(1 - \xi)/(1 + \xi)$ (Banjai 2010) and (2.11) is $\sum_{j=0}^\infty \phi_j(t) \xi^j = e^{-2t} f(\xi)$, where $f(\xi) = \exp(4t/(1 + \xi))$. This gives $f(\xi) = \sum_{j=0}^\infty f_j \xi^j$ where

$$f_j = \frac{1}{j!} \frac{d^j}{d\xi^j} f(\xi) \Big|_{\xi=0} = \frac{1}{j! (4t)^j} \frac{d^j}{dz^j} e^{-1/z} \Big|_{z=1/(4t)}$$

using the change of variables $z = (1 + \xi)/(4t)$. It follows from Olver, Lozier, Boisvert & Clark (2010, eq. 18.5.6) that $f_j = (-1)^j e^{-4t} L_j^{-1}(4t)$, where $L_j^\alpha(x)$ is a Laguerre polynomial. The identity $L_j^{-1}(x) = L_j(x) - L_{j-1}(x)$ (Gradshteyn & Ryzhik 1994, eq. 8.971–5) gives the trapezoidal rule basis functions $\phi_j(t) = (-1)^j \{\ell_j(4t) - \ell_{j-1}(4t)\}$, where $\ell_j(x) = e^{-x/2} L_j(x)$ is the j th Laguerre function. They are oscillatory, but do satisfy $\int_0^\infty \phi_j(t) dt = 1$. These low order basis functions are illustrated in Figure 1 below (see also Banjai (2010, Fig. 1) and Monegato et al. (2011, Fig. 4)).

Scheme	Initial $\phi_{-n} \equiv 0, n \geq 1$	Recurrence for basis functions $j \geq 1$
BDF1	$\phi_0(t) = e^{-t}$	$j\phi_j(t) - t\phi_{j-1}(t) = 0$
BDF2	$\phi_0(t) = e^{-3t/2}$	$j\phi_j(t) - 2t\phi_{j-1}(t) + t\phi_{j-2}(t) = 0$
BDF3	$\phi_0(t) = e^{-11t/6}$	$j\phi_j(t) - 3t\phi_{j-1}(t) + 3t\phi_{j-2}(t) - t\phi_{j-3}(t) = 0$
BDF4	$\phi_0(t) = e^{-25t/12}$	$j\phi_j(t) - 4t\phi_{j-1}(t) + 6t\phi_{j-2}(t) - 4t\phi_{j-3}(t) + t\phi_{j-4}(t) = 0$
Trap. rule	$\phi_0(t) = e^{-2t}$	$j\phi_j(t) - 4t\phi_{j-1}(t) + 2(j-1)\phi_{j-1}(t) + (j-2)\phi_{j-2}(t) = 0$

Table 1: Recurrence relations for the CQ basis functions.

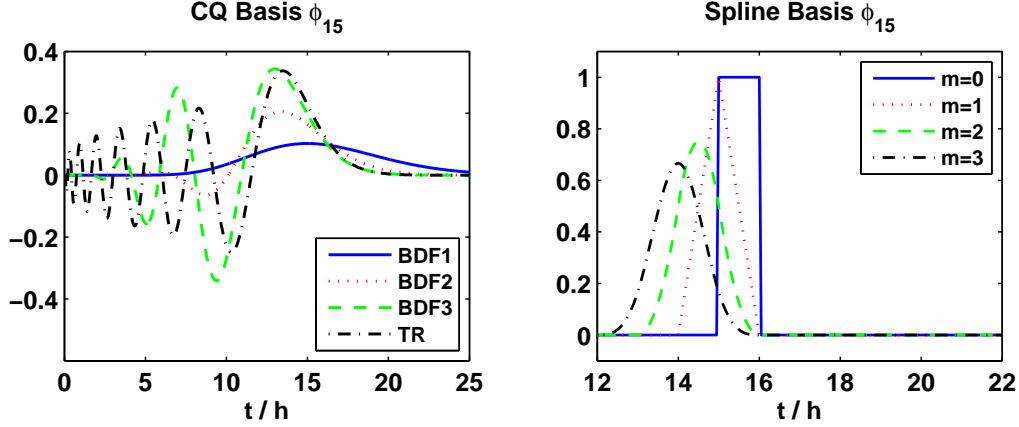


Figure 1: Typical CQ and spline basis functions. See Sections 2.3 and 4.1 for details.

The direct approach appears intractable for more complicated schemes (even for BDF3), and recurrence relations for the basis functions are given in Monegato et al. (2011, Sec. 3.2). They can be compactly derived by formally differentiating the generating function (2.11) with respect to ξ to get

$$\sum_{j=1}^{\infty} j\phi_j(t) \xi^{j-1} + t\delta'(\xi) \sum_{j=0}^{\infty} \phi_j(t) \xi^j = 0,$$

and then collecting terms in ξ . The initial conditions are $\phi_n(t) \equiv 0$ for $n < 0$, and the first term of the Taylor expansion of (2.11) gives $\phi_0(t) = e^{-\delta(0)t} = e^{-\delta_0 t}$. Recurrence relations for BDF1–4 and the trapezoidal rule are given in Table 1.

3 Convolution spline approach

As discussed in Sec. 1, basis functions with global support (such as those described above) give rise to dense matrices \mathbf{Q}^m in the TDBIE scheme (1.3), and this has storage and computational cost implications. Here we explore the use of compactly supported basis functions, which although not derived via standard CQ, nevertheless do fit into the CQ form (2.9). We set up a general framework for basis functions for the VIE (1.2) which are (mainly) translates, and consider specific examples based on B-splines in Secs. 4–5. This new approach gives sparse system matrices when used to time-step TDBIEs, and results are presented in Sec. 6.

3.1 Construction of a convolution spline scheme for (1.2)

We consider approximations of the form (2.9), but where all the basis functions ϕ_j have **compact support** of width $\mathcal{O}(h)$ and almost all are **translates** of a standard, compactly supported basis function ϕ_m , i.e.

$$\phi_j(t/h) = \phi_m(t/h + m - j) \quad \text{for } j \geq m. \quad (3.1)$$

The translate property (3.1) means that the approximation $U(t_n - t) \approx u(t_n - t)$ has the form

$$U(t_n - t) = \sum_{j=0}^{m-1} v_{n-j} \phi_j \left(\frac{t}{h} \right) + \sum_{j=m}^n v_{n-j} \phi_m \left(\frac{t}{h} + m - j \right) \quad (3.2)$$

for $t \geq 0$, where v_j approximates $u(t)$ for t near (but not necessarily at) t_j , and a sum is defined to be zero if its upper index is less than its lower index. Substituting the approximation (3.2) into the integral equation (1.2) and collocating at each time level as described in Section 2.2 gives

$$\sum_{j=0}^n q_j v_{n-j} = a(t_n) \quad (3.3)$$

for $n = 0 : N$ where the weights q_j are defined by (2.10). The unknown coefficients $\{v_j\}_{j=0}^N$ are then found by time marching as in (2.7).

3.2 Stability of (3.3)

For TDBIE applications and analysis (see e.g. Davies & Duncan (2004)), we require the scheme (3.3) to be stable in the following sense, independent of the input function $a(t)$.

Definition 3.1 (Stability). *The scheme (3.3) is said to be stable when the impulse response sequence $\{p_n\}$ defined by*

$$p_0 = 1, \quad p_n = \frac{-1}{q_0} \sum_{j=1}^n q_j p_{n-j} \quad \text{for } n \geq 1 \quad (3.4)$$

satisfies $|p_n| \leq C$ for all n such that $nh \leq T$, where the constant C is independent of h .

This is weaker than BIBO (bounded input bounded output) stability in the signal processing literature (see e.g. Proakis & Manolakis (2007)), which requires boundedness of the absolute sum $\sum_{n=0}^{\infty} |p_n| < \infty$.

Stability properties of the scheme (3.3) can be established by using the Z-transform, defined as follows.

Definition 3.2. *The Z-transform of a sequence $\{f_n\}_{n=0}^{\infty}$ is the function F given by*

$$F(\xi) = \mathcal{Z}\{f_n\}(\xi) = \sum_{n=0}^{\infty} f_n \xi^n \quad (3.5)$$

where $\xi \in \mathbb{C}$ with $|\xi| \leq 1$ is such that the sum converges.

The Z-transform of the scheme (3.3) is

$$Q(\xi) V(\xi) = A(\xi), \quad (3.6)$$

where

$$Q(\xi) = \sum_{j=0}^{\infty} \xi^j \int_0^{\infty} K(t) \phi_j(t/h) dt \quad (3.7)$$

and we take $a_n = a(t_n)$. Similarly, the Z-transform of (3.4) is $Q(\xi)P(\xi) = q_0$, and so $P(\xi) = q_0/Q(\xi)$. We now state a sufficient condition for stability when $Q(\xi)$ is a rational polynomial.

Theorem 3.1 (Root condition for stability). *If the Z-transform $Q(\xi)$ of $\{q_n\}$ is a rational polynomial in ξ , then the approximation (3.3) is stable in the sense of Definition 3.1 if the roots ξ_k of $Q(\xi)$ satisfy the following for any constant $c \geq 0$ (independent of h): $|\xi_k| \geq 1/(1+ch)$ and any with $1/(1+ch) \leq |\xi_k| \leq 1$ are simple.*

Although this result is a variant of the “root condition” familiar (after the change of variable $z = 1/\xi$) from zero stability analysis of numerical methods for ODEs, it does not appear to have previously been derived or used to determine the stability of VIE schemes. Note that simple roots with $|\xi_k| = 1/(1+ch)$ make a bounded contribution to p_n as n increases by the standard result

$$|\xi_k|^{-n} = (1+ch)^n \leq e^{cT}$$

for $t_n \leq T$, but roots of this size with multiplicity $\mu \geq 2$ grow like $n^{\mu-1}$ and hence violate the stability definition.

Verifying the stability condition directly or via the root condition above for a general approximation scheme for (1.2) may be very complicated. But as we show below, schemes with the translate property (3.1) can be tackled within the framework of Laplace transforms originally introduced for CQ, and this approach gives a way to extend the scope of stability analysis to a far broader range of kernel functions.

Substituting the Laplace inversion formula for K into (2.10) gives $q_j = \frac{h}{2\pi i} \int_{\gamma} \overline{K}(s) \Phi_j(-sh) ds$, where $\Phi_j(s)$ is the Laplace transform of ϕ_j . Hence the approximation scheme (3.3) can be written as

$$a(t_n) = \frac{1}{2\pi i} \int_{\gamma} \overline{K}(s) y_n(sh) ds \quad (3.8)$$

where $y_n(sh) = h \sum_{j=0}^n v_{n-j} \Phi_j(-sh)$. We note that y_n plays the same role here that the approximate solution of the ODE (2.3) does in standard CQ.

The translate property (3.1) and the compact support of ϕ_m implies

$$\Phi_j(-sh) = e^{sh(j-m)} \Phi_m(-sh) \quad \text{for } j \geq m$$

and so

$$y_n(sh) - e^{sh} y_{n-1}(sh) = h v_n \Phi_0(-sh) + h \sum_{j=1}^m v_{n-j} (\Phi_j(-sh) - e^{sh} \Phi_{j-1}(-sh)) ,$$

(using $v_j \equiv 0, j \leq 0$). Taking the Z-transform of this expression gives $Y(\xi, sh) = h B(\xi, sh) V(\xi)/(1 - e^{sh}\xi)$ when $\xi \neq e^{-sh}$, where

$$B(\xi, sh) = \Phi_0(-sh) + \sum_{j=1}^m [\Phi_j(-sh) - e^{sh} \Phi_{j-1}(-sh)] \xi^j . \quad (3.9)$$

It hence follows from (3.8) that

$$A(\xi) = V(\xi) \frac{h}{2\pi i} \int_{\gamma} \overline{K}(s) \left(\frac{B(\xi, sh)}{1 - e^{sh}\xi} \right) ds$$

and comparison with (3.6) yields the alternative representation for the Z-transform of the weights q_j :

$$Q(\xi) = \frac{h}{2\pi i} \int_{\gamma} \overline{K}(s) \left(\frac{B(\xi, sh)}{1 - e^{sh}\xi} \right) ds . \quad (3.10)$$

The expression $B(\xi, sh)/(1 - e^{sh}\xi)$ plays a role similar to that of $(\delta(\xi)/h - s)^{-1}$ in standard CQ analysis, and it is the key quantity in determining whether the scheme is stable or not. Unfortunately it has a more complicated structure: it has an infinite vertical line of simple poles at $s = s_k$ for $k \in \mathbb{Z}$, where

$$s_k := \frac{1}{h} (-\ln |\xi| - i \operatorname{Arg}(\xi) + i 2\pi k) \quad (3.11)$$

and the principal argument $\operatorname{Arg}(\xi) \in (-\pi, \pi]$. Note that if $|\xi| < 1$ then $\operatorname{Re}(s_k) > 0$.

To evaluate $Q(\xi)$ defined by (3.10) for a given kernel function $K(t)$ we can use either the left or right “D” contours illustrated in Figure 2, taking the limit $R \rightarrow \infty$ and setting $\gamma = \lim_{R \rightarrow \infty} \gamma_R$. Using the right contour gives

$$Q(\xi) = \sum_{k=-\infty}^{\infty} \overline{K}(s_k) B(\xi, s_k h) - \lim_{R \rightarrow \infty} \frac{h}{2\pi i} \int_{C_R^+} \overline{K}(s) \left(\frac{B(\xi, sh)}{1 - e^{sh}\xi} \right) ds .$$

The integral round C_R^+ does not necessarily vanish as $R \rightarrow \infty$ since for some basis functions (including higher order B-splines) the quantity $\frac{B(\xi, sh)}{1 - e^{sh}\xi} = \mathcal{O}(e^{csh})$ as $\operatorname{Re}(s) \rightarrow \infty$ for $c \geq 1$. We may also use the left contour when $\overline{K}(s)$ has simple poles at $s = \kappa_j$ with $\operatorname{Re}(\kappa_j) \leq 0$ and obtain the analogous result

$$Q(\xi) = h \sum_j \frac{B(\xi, \kappa_j h)}{1 - e^{\kappa_j h} \xi} \lim_{s \rightarrow \kappa_j} (s - \kappa_j) \overline{K}(s) - \lim_{R \rightarrow \infty} \frac{h}{2\pi i} \int_{C_R^-} \overline{K}(s) \left(\frac{B(\xi, sh)}{1 - e^{sh}\xi} \right) ds .$$

The asymptotic behaviour of the integral C_R^- as $R \rightarrow \infty$ is determined primarily by $\overline{K}(s)$. The extension of this left contour approach to poles with higher multiplicity is straightforward. We illustrate the use of these formulae in Sec. 4.2 for various kernels K when the basis functions are B-splines.

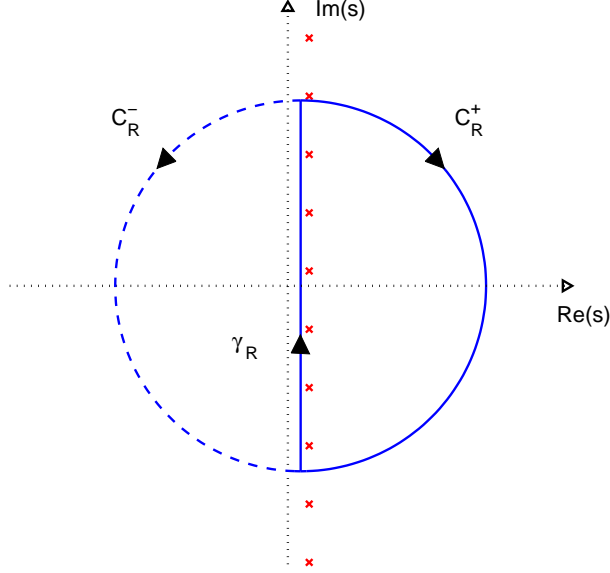


Figure 2: The left and right “D” contours of radius R used for the stability and Z-transform calculations. The crosses are the poles (3.11) and the vertical line of length (approximately) $2R$ is γ_R .

4 B-spline basis functions for (1.2)

We now illustrate the theoretical framework introduced in Section 3 for basis functions ϕ_j which are B-splines on $[0, \infty)$. We begin by listing some general properties of B-splines which are needed in the subsequent analysis, and then examine the stability of the convolution spline approximation of (1.2) for different example kernels. We also prove that the approximation given by (3.3) converges to the solution u of (1.2) for general smooth a and K . The convergence rate is at most second order, no matter how high the polynomial degree, because quasi-interpolation by the Schoenberg B-spline operator is at most $\mathcal{O}(h^2)$ (De Boor 1978). However a simple modification of the B-spline basis near $t = 0$ can give higher order stable approximations of (1.2), and this is analysed for the cubic case in Section 5.

4.1 Notation and properties

Throughout this section we assume that the basis functions $\{\phi_j\}_{j \geq 0}$ are (iso-geometric) B-splines of polynomial degree m based on the nodes (or knots) t_j for $j \geq 0$. We assume uniform spacing, so $t_{j+1} - t_j = h$ for $j \geq 0$. It is necessary for the B-spline basis functions to have the sum to unity property (2.12) in the whole interval $[0, \infty)$, and we introduce m new knots $t_j = 0$ for $j = -m : -1$. The m th degree B-splines are $b_j^m(t)$ for $j \geq -m$, and B-splines of degree $m > 0$ are recursively defined in terms of those of lower degree as follows, using the convention that $b_j^m(t) \equiv 0$ for $j < -m$.

Definition 4.1. (De Boor 1978) When $m = 0$

$$b_j^0(t) = \begin{cases} 1 & \text{if } t \in [t_j, t_{j+1}) \text{ for } j \geq 0 \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

If $m > 0$ then

$$b_j^m(t) = \left(\frac{t - t_j}{t_{j+m} - t_j} \right) b_j^{m-1}(t) + \left(\frac{t_{j+m+1} - t}{t_{j+m+1} - t_{j+1}} \right) b_{j+1}^{m-1}(t)$$

where the convention is that $0/0$ is interpreted as 0.

We shall make use of several B-spline properties in Section 4 (see for example standard references such as De Boor (1978), Schumaker (2007)), which we list here for convenience.

B-spline properties

- P1. Compact support. $b_j^m(t) = 0$ outwith $[t_j, t_{j+m+1})$, and $b_j^m(t_j) = 0$ unless $j = -m$.

P2. Translate property. If $j \geq 0$ then $b_j^m(t) = b^m(t/h - j)$, where the functions b^m are defined recursively:

$$b^0(\tau) = \begin{cases} 1 & \text{if } \tau \in [0, 1), \\ 0 & \text{otherwise} \end{cases} \quad \text{and if } m \geq 1: \quad b^m(\tau) = \frac{\tau}{m} b^{m-1}(\tau) + \frac{m+1-\tau}{m} b^{m-1}(\tau-1).$$

It follows that $\phi_j(\tau) = b^m(\tau + m - j)$ for $j \geq m$.

P3. Sum to unity. $\sum_{j=-m}^{\infty} b_j^m(t) = 1$ for all $t \geq 0$.

P4. Moments.

$$\int_{t_{j-m}}^{t_{j+1}} b_{j-m}^m(t) dt = \frac{t_{j+1} - t_{j-m}}{m+1} \quad \text{and} \quad \int_{t_{j-m}}^{t_{j+1}} t b_{j-m}^m(t) dt = \frac{t_{j+1} - t_{j-m}}{(m+1)(m+2)} \sum_{k=0}^{m+1} t_{j-m+k}.$$

P5. Shoenberg quasi-interpolation. Suppose that $m \geq 1$ and set $t_j^m = \frac{h(m+j)(m+j+1)}{2m}$ for $j = -m : -1$

and $t_j^m = t_{j+(m+1)/2}$ for $j \geq 0$. Then $\sum_{j=-m}^{\infty} t_j^m b_j^m(t) = t$ when $t \geq 0$.

It follows from properties P1, P3 and P5 above that

$$f(t) = \sum_{j=-m}^{\infty} f(t_j^m) b_j^m(t) + \mathcal{O}(h^2) \quad (4.1)$$

for any $f \in C^2[0, \infty)$, and if $f \in C^{p+1}[0, \infty)$ for $p \geq 2$ and $t \in [t_\ell, t_{\ell+1})$ for some $\ell \geq 0$, then

$$f(t) - \sum_{j=\ell-m}^{\ell} f(t_j^m) b_j^m(t) = \sum_{k=2}^p \frac{f^{(k)}(t_\ell)}{k!} \left[(t - t_\ell)^k - \sum_{j=\ell-m}^{\ell} (t_j^m - t_\ell)^k b_j^m(t) \right] + \mathcal{O}(h^{p+1}). \quad (4.2)$$

Throughout this section we shall use basis functions

$$\phi_j(t/h) = b_{j-m}^m(t) \quad \text{for } j \geq 0, \quad (4.3)$$

and it follows from P1 that the CQ weights are

$$q_j = \int_{t_{j-m}}^{t_{j+1}} K(t) b_{j-m}^m(t) dt. \quad (4.4)$$

The convergence analysis relies crucially on knowing the values of the weights when K is a constant, and this follows immediately from P4: when $K \equiv 1$ the weights q_j of (4.4) are given by

$$\frac{q_j}{h} = \begin{cases} \frac{j+1}{m+1} & \text{for } j = 0 : m-1 \\ 1 & \text{if } j \geq m. \end{cases}$$

4.2 Stability results for convolution B-splines

We now use the theoretical framework introduced in Section 3 to examine the stability of the convolution B-spline approximation of (1.2) for different example kernels which capture some of the important properties of TDBIE problems. These are: $K(t)$ equal to a constant, a step function, and the highly oscillatory kernels $K(t) = J_0(\omega t)$ or $\cos(\omega t)$, where ω can be of the order of $1/h$. We use $B_m(\xi, sh)$ to denote the function defined by (3.9) for the degree m basis functions, and $Q_m(\xi)$ to denote the coefficient Z-transform given by (3.10). The first few values of $B_m(\xi, sh)$ are listed in Table 2; those for higher values of m are more complicated, but are easily computed in a standard algebraic manipulation package.

In three of the cases $Q_m(\xi)$ is a rational polynomial in ξ and Theorem 3.1 can be used to determine stability. The Bessel function case is more complicated, and stability is determined from the Z-transform inversion formula by bounding the coefficients p_n of (3.4) directly. Note that this bound is independent of n , and so is a practically useful stability result, in contrast with the (essentially) uncheckable hypotheses needed in Davies & Duncan (2003).

B-spline degree	$B_m(\xi, s)$	$\lim_{s \rightarrow 0} B_m(\xi, s)$	$B_m(e^{-s}, s)$
$m = 0$	$s^{-1}(e^s - 1)$	1	$s^{-1}(e^s - 1)$
$m = 1$	$s^{-2}(e^s(\xi s - \xi + 1) + (\xi - s - 1))$	$(1 + \xi)/2$	$s^{-2}(e^s - 1)^2 e^{-s}$
$m = 2$	$s^{-3}(e^{2s}(\xi^2 - \xi) + e^s(2 - 2\xi^2 s + \xi(s^2 + 2s - 2)) + ((2s + 3)\xi - s^2 - 2s - 2 - \xi^2))$	$(1 + \xi + \xi^2)/3$	$s^{-3}(e^s - 1)^3 e^{-2s}$
$m = 3$	\dots	$(1 + \xi + \xi^2 + \xi^4)/4$	$s^{-4}(e^s - 1)^4 e^{-3s}$

Table 2: The function $B_m(\xi, s)$ for $m = 0 : 3$. See text for details.

4.2.1 Constant kernel: $K(t) = 1$, transform $\bar{K}(s) = 1/s$

Integrating (3.10) round the left contour in Figure 2 gives

$$Q_m(\xi) = h \lim_{s \rightarrow 0} \frac{B_m(\xi, sh)}{1 - e^{sh}\xi} = \frac{h(1 - \xi^{m+1})}{(m+1)(1 - \xi)^2}. \quad (4.5)$$

The function Q_m has m simple roots on the unit circle, and stability of the approximation then follows from Theorem 3.1. (Note that stability also follows from the convergence result of Sec. 4.3.)

4.2.2 Discontinuous step-function kernel: $K(t) = 1$ for $t \in [0, L]$, otherwise 0.

Discontinuous kernels can arise in TDBIE problems, even when the scattering surface Γ is smooth and closed. Examples (in Laplace transformed representation) are given in Banjai & Lubich (2011, Sec. 6.1) and Sauter & Veit (2011b, Sec. 4.1) describing time domain scattering where only the zeroth order harmonic in space is excited on the surface of a sphere. Similar, but more complicated discontinuous kernels are described in Sauter & Veit (2011b) for more general scattering from spheres involving higher spatial harmonics.

We assume that the duration L is independent of h and denote the integer part of L/h by M , i.e. when h is sufficiently small, $L = (M + r)h$ for integer $M > m$ and $r \in [0, 1)$. It is simplest to work with the explicit Z-transform formula (3.5) using the weights given in (4.4). Results for $m = 0 : 3$ are summarised below.

Case $m = 0$:
$$\frac{Q_0(\xi)}{h} = r\xi^M + \sum_{n=0}^{M-1} \xi^n = \frac{1}{\xi - 1} (r\xi^{M+1} + (1 - r)\xi^M - 1).$$

When $r \in (0, 1)$ it can be shown that the M roots ξ_j of Q_0 satisfy $|\xi_j| > 1$ for $j = 1 : M$, and when $r = 0$ there are $M - 1$ simple roots $\xi_j = \exp(i2\pi j/M)$ for $j = 1 : M - 1$. Hence Theorem 3.1 implies that the $m = 0$ scheme is stable for all L .

Case $m = 1$:

$$\frac{2}{h} Q_1(\xi) = \begin{cases} \frac{(1 + \xi)(1 - \xi^M)}{(1 - \xi)} & \text{when } r = 0 \\ r^2 \xi^{M+1} + (2r - r^2 + 1)\xi^M + 2 \sum_{n=1}^{M-1} \xi^n + 1 & \text{for } r \in (0, 1). \end{cases}$$

When $r = 0$ the roots of Q_1 are -1 and $\exp(i2\pi j/M)$ for $j = 1 : M - 1$, so there is a double root when M is even, giving a linear (and not exponential) instability. When $r > 0$ the modulus of the product of the $M + 1$ roots of $Q_1(\xi)$ is $1/r^2$ and it can be shown that there is a real negative root with modulus strictly greater than $1/r^2$, implying that one or more of the roots has $|\xi_j| < 1$ and the scheme is unstable.

In summary, the $m = 1$ scheme is unstable unless L/h is an odd integer, which is not always possible.

Case $m = 2 : 3$. Similar arguments to those above can be used to show that the scheme is unstable for any r in these cases. Although note that the modified cubic spline basis functions described in Section 5 give completely stable results for this kernel.

4.2.3 $K(t) = J_0(\omega t)$, transform $\bar{K}(s) = 1/\sqrt{s^2 + \omega^2}$

This is the kernel function that arises when considering TDBIE scattering from the flat surface \mathbb{R}^2 , where ω can be of the order of $1/h$ (i.e. $h\omega$ is bounded as $h \rightarrow 0$, but does not necessarily tend to zero). Its Laplace transform has a branch cut between the values $s = \pm i\omega$, and the Z-transform $Q_m(\xi)$ of the weights is not a

rational polynomial. We can still establish stability directly for the impulse response sequence $\{p_n\}$ defined in (3.4) using a change of variable in the Z-transform inversion formula (Doetsch 1971, eq. 37.7) to get

$$p_n = \frac{e^{nh\sigma} q_0}{2\pi} \int_{-\pi}^{\pi} \frac{e^{iny}}{Q_m(e^{-x-iy})} dy, \quad (4.6)$$

where we have set $\xi = e^{-sh}$ with $s = \sigma + i\eta$ and $\sigma > 0$ and then changed to scaled variables $x = \sigma h$ and $y = \eta h$. This yields the bound

$$|p_n| \leq \frac{e^{\sigma T}}{2\pi} \int_{-\pi}^{\pi} \frac{|q_0| dy}{|Q_m(e^{-x-iy})|} \quad (4.7)$$

when $t_n \leq T$, which holds for any fixed $\sigma > 0$ when the singularities of the integrand are to the left of x . Note that this bound is independent of n , and the scheme is stable at a given frequency ω if the integral term in (4.6) remains bounded as $h \rightarrow 0$. This can be demonstrated using the right contour in Fig. 2 to calculate $Q_m(e^{-sh})$ but it is more straightforward to work directly with (3.7).

It follows from standard properties of the B-spline basis (4.3) that

$$q_0 = \int_0^h \left(1 - \frac{t}{h}\right)^m J_0(\omega t) dt = \frac{m! K^{(-m-1)}(h)}{h^m}$$

where functions $K^{(-k)}(t)$ are recursively defined by

$$K^{(0)}(t) = J_0(\omega t), \quad K^{(-k-1)}(t) = \int_0^t K^{(-k)}(t') dt' \quad \text{for } k = 0, 1, \dots$$

Note that

$$K^{(-m-1)}(t) \leq t^{m+1}/(m+1)! \quad (4.8)$$

for all $t \geq 0$.

Properties of the B-spline basis functions can also be exploited to write (3.7) as

$$Q_m(\xi) = \frac{(1-\xi)^{m+1}}{\xi h^m} \mathcal{Z}\{K^{(-m-1)}\}(\xi) + C_m(\xi) \quad (4.9)$$

where the ‘‘correction’’ terms are $C_0 = 0$, $C_1 = 0$,

$$C_2(\xi) = (1-\xi) \frac{K^{(-3)}(h)}{h^2}, \quad C_3(\xi) = (1-\xi)(5-3\xi) \frac{K^{(-4)}(h)}{h^3} + \xi(1-\xi) \frac{K^{(-4)}(2h)}{2h^3}.$$

The presence of these terms is because for $m = 0 : 1$ the basis functions are pure translates, while for $m \geq 2$, there are different shaped basis functions at the start. The function $K^{(-k)}$ has Laplace transform

$$\overline{K^{(-k)}}(s) = \frac{1}{s^k \sqrt{s^2 + \omega^2}}$$

and it follows from the Poisson sum formula relating Z and Laplace transforms that

$$\mathcal{Z}\{K^{(-k)}\}(e^{-sh}) = \frac{1}{h} \sum_{j \in \mathbb{Z}} \frac{1}{s_j^k \sqrt{s_j^2 + \omega^2}} := h^k \sum_{j \in \mathbb{Z}} f_j^{k-1}$$

where $s_j = s + i2\pi j/h$, and we use this expression in (4.9) in order to bound the integral term in (4.7).

When $m = 0$ it is possible to obtain an analytic bound when $\omega \leq \pi/h$, and a careful numerical approximation of the integral (4.7) indicates that the p_n are bounded for ω up to (at least) $20\pi/h$. The situation is more complicated for $m = 1 : 3$, and in these cases we give numerical bounds.

Case $m = 0$: From (4.8) and (4.9)

$$\frac{|q_0|}{|Q_0(e^{-x-iy})|} = \frac{1}{|e^{x+iy} - 1| |\sum_{k \in \mathbb{Z}} f_k^0|}$$

and

$$\left| \sum_{k \in \mathbb{Z}} f_k^0 \right| = |f_0^0| \left(\left(1 + \sum_{k \in \mathbb{Z}/0} \Re(f_k^0/f_0^0) \right)^2 + \left(\sum_{k \in \mathbb{Z}/0} \Im(f_k^0/f_0^0) \right)^2 \right)^{1/2} \geq |f_0^0| \left| 1 + \sum_{k \in \mathbb{Z}/0} \Re(f_k^0/f_0^0) \right|.$$

It can be shown that

$$1 + \min_y \sum_{k \in \mathbb{Z}/0} \Re(f_k^0/f_0^0) = 1 + \sum_{k \in \mathbb{Z}/0} \Re(f_k^0/f_0^0)|_{y=0} > \frac{2}{3}$$

when $0 \leq x \leq 1$, $0 \leq \omega h \leq \pi$ and $|y| \leq \pi$. In this case

$$\frac{|q_0|}{|Q_0(e^{-x-iy})|} \leq \frac{3\sqrt{x^2+2\pi^2}}{2} \frac{\sqrt{x^2+y^2}}{|e^{x+iy}-1|} \leq \frac{3\sqrt{x^2+2\pi^2}}{2} \frac{\pi e^{-x/2}}{2}$$

using Jordan's inequality. Together with (4.7) this proves that the scheme is stable in the sense of Definition 3.1 for frequency ω in the contiguous interval $0 \leq \omega h \leq \pi$. Numerical evaluation of the right hand side of (4.7) indicates that the bound is

$$|p_n| \leq 1.3 e^{\sigma T}$$

when h is sufficiently small (so that $x < 0.1$) and $0 \leq \omega h \leq 20\pi$. Further numerical tests computing p_n directly from (3.4) for a finite number of steps $n \leq 2500$ and the same range of values of ωh indicate that $|p_n| \leq 1$, consistent with the estimate above. There is no indication of instability at any value of ωh tested and we speculate that this scheme is stable for all ω .

Case m = 1: Finding an explicit bound for the integral in (4.7) is significantly more complicated and perhaps even intractable here so we only consider its direct numerical evaluation over a range of frequencies and values of $x = h\sigma$ close to 0. However there is an extra complication because $q_0/Q_1(z)$ has a pole at $z = -1$. This is most obvious when we set $\omega = 0$ and get $q_0/Q_1(z) = (1-z)/(1+z)$ from (4.5). When $\omega \neq 0$ there is no simple formula, but it is still possible to show by direct evaluation of the summation formula for $Q_1(e^{-x-iy})$ that the pole remains when $0 < \omega h < \pi$. The pole renders the bound in (4.7) less useful since

$$|q_0| \int_{-\pi}^{\pi} \frac{dy}{|Q(e^{-x-iy})|} = \mathcal{O}(\log(1/x)) \rightarrow \infty \quad \text{as } x \rightarrow 0$$

(where $x = h\sigma$) and hence $|p_n| \leq (C_0 + C_1 \log(1/h))e^{\sigma T}$ as $h \rightarrow 0$, which does not satisfy the stability requirement of Def. 3.1. Fortunately the singularity can be removed by writing

$$\frac{q_0}{Q_1(\xi)} = \frac{a}{(1+\xi)} + \Delta P(\xi), \quad \text{where } a = \lim_{\xi \rightarrow -1} \frac{q_0(1+\xi)}{Q_1(\xi)}$$

so that $\Delta P(\xi)$ is bounded as $\xi \rightarrow -1$. The sequence $\{p_n\}$ can then be written as $p_n = a(-1)^n + \Delta p_n$ where Δp_n is bounded in the same way as (4.7):

$$|\Delta p_n| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |\Delta P(x+iy)| dy. \quad (4.10)$$

Numerical evaluation of the integral over frequencies $0 \leq \omega h < \pi$ indicates that $|\Delta p_n| \leq 1.1$ and $0 < a \leq 2$ for $nh \leq T$ and $0 < x \leq 1/10$. Combining this with direct evaluation of (4.7) when $\omega h \in [\pi, 20\pi]$ and there is not a pole indicates that

$$|p_n| \leq C e^{\sigma T} \quad \text{where } C = \begin{cases} 3.1, & \omega h \in [0, \pi) \\ 1.1, & \omega h \in [\pi, 20\pi] \end{cases}$$

for $nh \leq T$ and $0 < x \leq 1/10$ satisfying the stability Def. 3.1. Further numerical tests computing p_n directly from (3.4) for a finite number of steps $n \leq 2500$ and the same range of values of ωh indicate that $|p_n| \leq 2$ for $\omega h \in [0, 0.7\pi)$ and $|p_n| \leq 1$ for $\omega h \in [0.7\pi, 20\pi]$, consistent with the estimate above. Again we speculate that this scheme is stable for all ω .

Case m = 2: The function $q_0/Q_2(\xi)$ appears to have two poles on the unit circle when $\omega h \in [0, L)$ where $L \approx 2.55$, symmetrically located at $\xi = e^{\pm i\mu(\omega h)}$. In the simple case $\omega = 0$, (4.5) gives $Q_2(\xi) = q_0(1+\xi+\xi^2)/(1-\xi)$, and so $\mu(0) = 2\pi/3$. Numerical evidence indicates that $\mu(\omega h) > \omega h$ and that μ increases until the two poles meet where $\mu(L) = \pi$. At that point stability in the sense of Def. 3.1 breaks down since there does not appear to be any compensating factor in the numerator to reduce the order of this double singularity.

We locate the poles numerically, and remove them from the integrand $q_0/Q_2(\xi)$ in a similar way to the previous case. The simplest form that captures the main features of the behaviour is

$$\frac{q_0}{Q_2(\xi)} = \frac{a(1-\xi)}{\xi^2 - 2\xi \cos \mu + 1} + \Delta P(\xi),$$

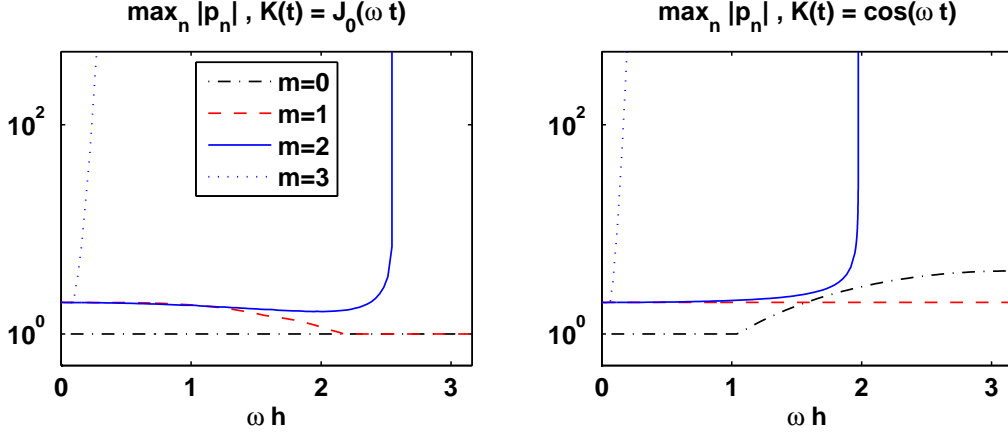


Figure 3: Plots of $\max\{|p_n| : 0 \leq n \leq 2500\}$ against ωh for the B-spline schemes with $m = 0 : 3$ applied to the highly oscillatory kernels $K(t) = J_0(\omega t)$ (left plot) and $K(t) = \cos \omega t$ (right plot). See text for more details.

so that by direct inversion of the Z transform

$$p_n = a(\cos(n\mu) - \sin(n\mu) \tan(\mu/2)) + \Delta p_n.$$

For $0 \leq \omega h < L \approx 2.55$ we find that $0 < a \leq 1$ and from (4.10) that $|\Delta p_n| \leq 0.8$, giving

$$|p_n| \leq 0.8 + \sec(\mu(\omega h)/2)$$

for $nh \leq T$ when $0 < x \leq 1/10$. This satisfies Def. 3.1 since $2\pi/3 \leq \mu(\omega h) < \pi$, but since $\sec(\mu/2) \rightarrow \infty$ as $\mu \rightarrow \pi$, the possibility for instability is clear. Further numerical tests computing p_n directly from (3.4) for a finite number of steps show very close and consistent agreement with this bound on $|p_n|$, with instability appearing as predicted at $\omega h = L \approx 2.55$ – i.e. there is a contiguous interval of stability $\omega h \in [0, L)$ with $L \approx 2.55$.

Case $m = 3$: Not surprisingly this case is more complicated still. When $\omega = 0$ the three poles of $q_0/Q_3(\xi)$ are on the unit circle at $\xi = -1, e^{\pm i\pi/2}$. However, when $\omega h > 0$ increases, the real-valued pole at $\xi = -1$ moves (harmlessly) outside the unit circle while the other complex conjugate pair moves inside causing instability. Numerical tests computing p_n directly from (3.4) for fixed values of ωh show behaviour consistent with this: we see apparent stability for larger values of h which disappears as $h \rightarrow 0$, i.e. this scheme is stable only when ω is fixed (so that $\omega h \rightarrow 0$).

Similar results can be proved for the (more straightforward) oscillatory kernel $K(t) = \cos \omega t$, as summarised below.

- **$m = 0$** : the scheme is stable at any frequency ω for which $\omega h \in [0, \pi)$;
- **$m = 1$** : the scheme is stable at any frequency ω for which $\omega h \in [0, 2\pi)$;
- **$m = 2$** : the scheme is stable at any frequency ω for which $\omega h \in [0, \theta)$, where $\theta = 1.9747 \dots$;
- **$m = 3$** : there is no $\mathcal{O}(1)$ interval of stability for ωh , but the scheme is stable for bounded ω .

The stability results for highly oscillatory kernels are illustrated in Figure 3. The plots show $\max_n |p_n|$ for $n = 0 : 2500$ for the B-spline schemes with $m = 0 : 3$ applied to the kernels $K(t) = J_0(\omega t)$ (left plot) and $K(t) = \cos \omega t$ (right plot). Over the range $\omega \in [0, \pi/h]$ shown, the general stability behaviour for these two kernels is similar. In particular the left plot illustrates the stability when $m = 0, 1$, while scheme $m = 2$ is stable for $\omega h \in [0, L)$ with $L \approx 2.55$. On the right plot, scheme $m = 0$ is stable except at $\omega h = \pi$, scheme $m = 1$ is stable and $m = 2$ scheme is stable for $\omega h \in [0, L)$ with $L \approx 1.97$. On both plots the $m = 3$ scheme is clearly unstable when $\omega = \mathcal{O}(1/h)$.

4.3 Convergence results for (3.3)

Formal convergence of a method when applied to a smooth VIE problem is certainly a necessary condition for it to behave well (for VIEs or TDBIEs), and we now prove that the collocation spline approximation (3.3) with B-spline basis functions (4.3) converges to the solution u of (1.2) for general smooth a and K . The analysis proceeds by considering the case $K \equiv 1$ and then using Taylor expansion to show that the same result also holds for smooth K with $K(0) = 1$ when h is small enough (see e.g. Brunner (2004)). Thus it does not apply to the important case of an oscillatory kernel where the oscillation frequency $\omega = \mathcal{O}(1/h)$, whose stability was analysed above for the Bessel function and cosine kernels.

When $m = 0$ the approximation (3.2) is the same as using piecewise constant collocation (at the interval endpoints), and this has been fully analysed (see e.g. Brunner (2004) for details). Here we assume that $m \geq 1$ (note that this includes the well-known case of piecewise linear approximations of (1.2)), and show that convergence is always second order, no matter how high the polynomial degree, because quasi-interpolation by the Schoenberg B-spline operator is at most $\mathcal{O}(h^2)$ (De Boor 1978). This is in marked contrast to discontinuous polynomial collocation or Galerkin approximations of (1.2) which converge at optimal order (Brunner 2004, Brunner, Davies & Duncan 2009, Brunner, Davies & Duncan 2012). However a simple modification of the B-spline basis near $t = 0$ can give higher order stable approximations of (1.2), as illustrated when $m = 3$ in Sec. 5.

The **approximation error** $e_n(t)$ for $n > 0$, $t \geq 0$ is

$$e_n(t) = u(t_n - t) - \sum_{j=0}^n v_{n-j} b_{j-m}^m(t), \quad (4.11)$$

where the coefficients v_j satisfy

$$a(t_n) = \sum_{j=0}^n q_j v_{n-j} \quad (4.12)$$

for weights q_j as defined in (4.4). Note that $v_0 = 0$ (because $a(0) = 0$) and so the sums above can be taken from $j = 0$ to $n - 1$, and it then follows from Property P1 that $e_n(t) = 0$ for $t \geq t_n$ and each weight can be written as $q_j = \int_0^{t_n} K(t) b_{j-m}^m(t) dt$. Hence, multiplying (4.11) by $K(t)$ and integrating gives

$$\int_0^{t_n} K(t) e_n(t) dt = \int_0^{t_n} K(t) u(t_n - t) dt - \sum_{j=0}^{n-1} q_j v_{n-j} = 0,$$

by (1.2) and (4.12), i.e. e_n is orthogonal to K on $(0, t_n)$. The formal convergence result is as follows.

Theorem 4.1. *Suppose that $m \geq 1$ and the conditions (1.5) and (2.1) hold for $d \geq 4$. Then*

$$|e_n(t)| \leq C h^2 \quad (4.13)$$

for $t \in [t_m, T]$, for some C independent of n and h . If $m = 1$ then (4.13) holds for $t \in [0, T]$.

Remarks:

- The $m = 1$ case has been fully analysed (Brunner 2004) and is just included for completeness.
- The restriction to second order convergence for $m > 1$ is a fundamental aspect of quasi-interpolation by classical B-splines and not an artefact of the proof, and is illustrated in Figure 4 when (2.1) holds with $d = 4$.
- Equation (4.13) trivially holds for $t \geq t_n$ (because $e_n(t) = 0$), and so it is enough to prove the result for $t \in [t_m, t_n]$ when $m > 1$, where $n \leq T/h$.

Proof. We first express $e_n(t)$ in terms of coefficients $\varepsilon_k := u(t_{k+(m-1)/2}) - v_k$. Substituting the quasi-interpolation result (4.1) with $f(t) = u(t_n - t)$ in the definition (4.11) of $e_n(t)$ gives

$$e_n(t) = \sum_{j=m}^n \varepsilon_{n-j} b_{j-m}^m(t) + \sum_{j=n+1}^{n+m} u(t_n - t_{j-m}^m) b_{j-m}^m(t) + \mathcal{O}(h^2),$$

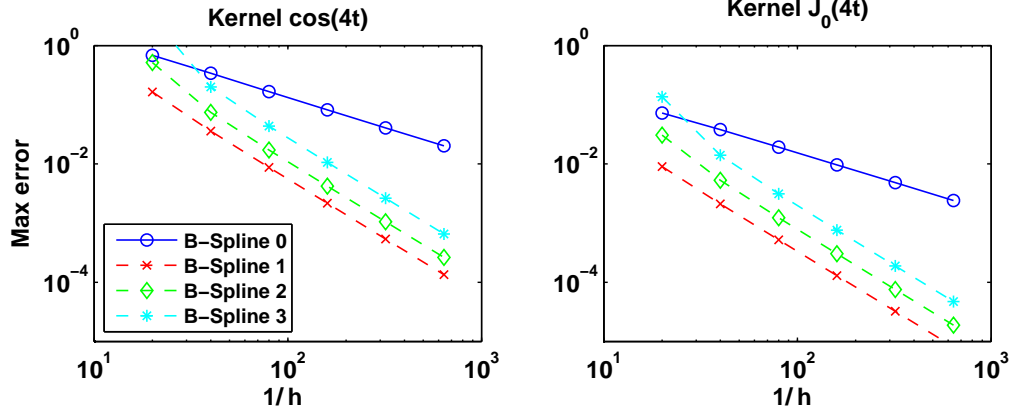


Figure 4: Convergence results for the approximation of (1.2) for two smooth kernel functions with maximum time $T = 10$ and $a(t) = t^6 \exp(-50(t - 1/2)^2)$. Convergence rates of $\mathcal{O}(h^2)$ for splines of degree $m \geq 1$ and $\mathcal{O}(h)$ for $m = 0$ are clear. Stability results for the highly oscillatory kernels $\cos \omega t$ and $J_0(\omega t)$ where the frequency ω can be $\mathcal{O}(1/h)$ are given in Section 4.2.3 and Figure 3.

where we have used $b_{j-m}^m(t) = 0$ for $j < m$ and $j > n + m$. It follows from the assumptions (1.5) and (2.1) that $u(ch) = \mathcal{O}(h^d)$ for any constant c . This implies that the second sum term in the previous equation is $\mathcal{O}(h^d)$, and hence yields

$$e_n(t) = \sum_{j=m}^n \varepsilon_{n-j} b_{j-m}^m(t) + \mathcal{O}(h^2)$$

for $t \in [t_m, t_n]$ with $t_n \leq T$. Because there are at most $m+1$ nonzero terms in this sum for any t , it is sufficient to show that there exists a constant C independent of h such that

$$|\varepsilon_j| \leq Ch^2 \quad \text{for all } j \leq T/h. \quad (4.14)$$

To prove (4.14) note that it follows from (4.12) that $\sum_{j=0}^n q_j v_{n-j} = \int_0^{t_n} K(t) u(t_n - t) dt$ and so

$$\sum_{j=0}^n \frac{q_j}{h} \varepsilon_{n-j} = \sum_{j=0}^n \frac{q_j}{h} u(t_{n-j+(m-1)/2}) - \frac{1}{h} \int_0^{t_n} K(t) u(t_n - t) dt := R_n. \quad (4.15)$$

If h is sufficiently small, then expanding $K(t)$ and using P4 gives

$$\frac{q_j}{h} = \begin{cases} \frac{j+1}{m+1} + h K'(0) \frac{(j+1)^2(j+2)}{2(m+1)(m+2)} + \mathcal{O}(h^2) & \text{for } j = 0 : m-1 \\ K(t_{j-m}) + \frac{1}{2} h (m+1) K'(t_{j-m}) + \mathcal{O}(h^2) & \text{if } j \geq m. \end{cases} \quad (4.16)$$

It then follows from the quasi-interpolation result (4.2) with $p = 3$ that when $n \geq m$,

$$h R_n = \sum_{\ell=0}^{n-1} \sum_{k=2}^3 \eta_\ell^k u^{(k)}(t_{n-\ell}) + \mathcal{O}(h^4) \quad \text{for } \eta_\ell^k = \frac{(-1)^{k+1}}{k!} \int_{t_\ell}^{t_{\ell+1}} K(t) \left\{ (t - t_\ell)^k - \sum_{j=\ell}^{\ell+m} (t_{j-m} - t_\ell)^k b_{j-m}^m(t) \right\} dt.$$

It is then straightforward to show $R_{n+1} - 2R_n + R_{n-1} = \mathcal{O}(h^3)$ and after some manipulation using (4.16) the second central difference of (4.15) can be written as

$$(1 + h \mu_o) \varepsilon_{n+m+1} = \varepsilon_n + h \sum_{\ell=0}^m \mu_{\ell+1} \varepsilon_{n-\ell+m} + h^2 \sum_{\ell=m+1}^n \mu_{\ell+1} \varepsilon_{n-\ell+m} + \gamma_n, \quad (4.17)$$

where $\gamma_n = \mathcal{O}(h^3)$ and all the μ_ℓ are bounded. This can be written as a one-step recurrence for the vector $\underline{\delta}^n \in \mathbb{R}^{m+1}$ with components $\delta_j^n \equiv \varepsilon_{n+j}$ for $j = 0 : m$. The recurrence is $\delta_j^{n+1} = \delta_{j+1}^n$ for $j = 0 : m-1$ with

δ_m^{n+1} given by (4.17), which gives the matrix–vector system

$$\underline{\delta}^{n+1} = (M + h W_0) \underline{\delta}^n + h^2 \sum_{\ell=0}^{n-1-m} W_{n-\ell} \underline{\delta}^{m+\ell} + \gamma_n \underline{e}^m \quad (4.18)$$

where $\underline{e}^m = (0, \dots, 0, 1)^T$, each matrix W_ℓ is bounded and $M \in \mathbb{R}^{(m+1) \times (m+1)}$ is the circulant matrix whose only nonzero components are $M_{m,0} = M_{j,j+1} = 1$ for $j = 0 : m-1$. The eigenvalues of M are the $(m+1)$ -th roots of unity, and are hence distinct. Following Brunner (1978) we note that M belongs to Ortega’s (1990, §1.3) Class M, and so there is a vector norm $\|\cdot\|_*$ on \mathbb{R}^{m+1} for which the induced matrix norm satisfies $\|M\|_* = \rho(M) = 1$. Taking this norm of (4.18) then implies that there is a constant C such that

$$\|\underline{\delta}^{n+1}\|_* \leq (1 + Ch) \|\underline{\delta}^n\|_* + Ch^2 \sum_{\ell=0}^{n+1-m} \|\underline{\delta}^{m+\ell}\|_* + Ch^3$$

and the top bound of (4.14) gives $\|\underline{\delta}^0\|_* \leq Ch^d$. Standard arguments can then be used to show that $\|\underline{\delta}^n\|_* \leq \nu_n$ where $\nu_0 = Ch^d$ and

$$\nu_{n+1} = (1 + Ch) \nu_n + Ch^2 \sum_{\ell=0}^n \nu_\ell + Ch^3 \quad \text{for } n \geq 0. \quad (4.19)$$

This has the solution $\nu_n = A_+ \lambda_+^n + A_- \lambda_-^n$ where $\lambda_\pm = 1 + \mathcal{O}(h)$ and $A_\pm = \mathcal{O}(h^2)$. Hence $\nu_n \leq C_1 h^2 e^{C_2 T}$ for $n \leq T/h$, which concludes the proof of (4.14) and hence (4.13). \square

5 Modified cubic spline basis functions for (1.2)

The results of the previous section illustrate that the convolution spline framework can be used to derive new VIE approximations and prove their stability in cases not covered by standard convergence analysis (such as discontinuous or highly oscillatory kernels), but the restriction to second order convergence for the B-spline basis (4.3) is not competitive with RK-based CQ methods (Banjai 2010, Banjai & Lubich 2011, Banjai et al. 2011). We now show how a slight modification of the $m = 3$ B-spline basis near $t = 0$ can yield methods which are more accurate and have better stability properties than (4.3).

5.1 Derivation

It is simpler to define the modified basis functions in terms of B-splines centred at zero, and we set $B(t) = b^3(t+2)$, so $\text{supp}(B) = (-2, 2)$. For $j \geq 3$ the basis functions are $\phi_j(t) = B(t-j)$, and we choose ϕ_j for $j = 0 : 2$ to satisfy the “parabolic runout” conditions at $t = 0$, namely

$$\phi_0(t) = B(t) + 3B(t+1), \quad \phi_1(t) = B(t-1) - 3B(t+1), \quad \phi_2(t) = B(t-2) + B(t+1).$$

This means that

$$\sum_{j=-1}^{\infty} j^r (\phi_j(t) - B(t-j)) = 0 \quad \text{for } t \in [0, \infty) \text{ for } r = 0 : 2, \quad (5.1)$$

where we set $\phi_{-1}(t) \equiv 0$, and in particular it ensures that quasi-interpolation in terms of $\{\phi_j\}_{j=0}^{\infty}$ is linearity-preserving on $[0, \infty)$. Weights q_j for the VIE (1.2) are given in terms of these basis functions by (2.10) and the approximation at $t = t_n$ is

$$U_n(t_n - t) = \sum_{j=0}^n v_{n-j} \phi_j(t/h), \quad (5.2)$$

where the v_j coefficients are given by (3.3). The individual coefficients can be directly obtained from $U_n(t - t_n)$ by introducing dual basis functions $\Phi_k(t)$ such that

$$\int_0^\infty \phi_j(t) \Phi_k(t) dt = \delta_{j,k}.$$

There are many ways in which a suitable dual basis can be chosen, and one possibility is to use continuous piecewise cubic functions on $[0, \infty)$ defined with respect to knots in \mathbb{N} . For $k \geq 3$, we set $\Phi_k(t) = \Phi(t - k)$

where Φ is the even continuous piecewise cubic function on $[-2, 2]$ which is zero at ± 2 and is C^1 at the interior knots $\pm 1, 0$ and satisfies

$$\int_{-2}^2 \Phi(t - \ell) B(t) dt = \delta_{0,\ell} \quad \text{for } \ell = 0 : 3,$$

and it is also possible to find suitable continuous piecewise cubics Φ_k with support in $[0, k+2]$ for $k = 0 : 2$. Calculating these dual basis functions in an algebraic manipulation package is straightforward, although their coefficients are messy and we omit their details. Once the Φ_k have been obtained it can be shown by Taylor expanding that for any sufficiently smooth function f ,

$$\int_{\max(0, k-2)}^{k+2} f(t_n - hs) \Phi_k(s) ds = f(t_{n-k}) - \frac{h^2}{6} f''(t_{n-k}) + \mathcal{O}(h^4). \quad (5.3)$$

Multiplying (5.2) by $\Phi_k(t/h)$ and integrating over $[0, \infty)$ gives

$$v_{n-k} = \int_{\max(0, k-2)}^{k+2} U_n(t_n - hs) \Phi_k(s) ds$$

and we now use this representation and (5.3) in order to obtain the best approximation of the solution u of (1.2) in terms of the basis functions ϕ_j when (1.5) holds with $d = 4$. Specifically, we set

$$u(t_n - t) = \sum_{j=0}^n u_{n-j} \phi_j(t/h) + \tilde{R}_n(t), \quad (5.4)$$

where

$$u_{n-j} := \int_{\max(0, j-2)}^{j+2} u(t_n - hs) \Phi_j(s) ds = u(t_{n-j}) - \frac{h^2}{6} u''(t_{n-j}) + \mathcal{O}(h^4). \quad (5.5)$$

We now show that this scheme is fourth order accurate, and then discuss its stability properties for discontinuous and highly oscillatory kernels.

5.2 Convergence

It follows from (5.4)–(5.5) that the approximation error $e_n(t) := u(t_n - t) - U_n(t_n - t)$ for this scheme is

$$e_n(t) = \sum_{j=0}^n \varepsilon_{n-j} \phi_j(t/h) + \tilde{R}_n(t)$$

where $\varepsilon_k = u_k - v_k$. We now show that modifying the basis functions near $t = 0$ as described above improves the scheme's accuracy from second to fourth order.

Theorem 5.1. *Suppose that the conditions (1.5) and (2.1) hold for $d \geq 6$. Then there exists a constant C independent of n and h such that if $t_n \leq T$, the VIE approximation error satisfies*

$$|e_n(t)| \leq C h^4 \quad (5.6)$$

for $t \in [t_1, t_n]$.

Proof. It is straightforward to verify that if $s \in [0, 1)$,

$$\tilde{R}_n(t_k + hs) = u(t_{n-k} - hs) - \sum_{\ell=-1}^2 u_{n-k-\ell} \phi_{k+\ell}(k+s)$$

where ϕ_{-1} is taken to be zero, and hence it follows from (5.1), (5.5) and standard B-spline properties that

$$\tilde{R}_n(t_k + hs) = \begin{cases} h^3(1-s)^3 u'''(t_n)/6 + \mathcal{O}(h^4) & \text{when } k = 0 \\ \mathcal{O}(h^4) & \text{when } k \geq 1. \end{cases}$$

It is thus sufficient to show that each $\varepsilon_k = \mathcal{O}(h^4)$ (because at most four basis functions are nonzero for any t). The proof follows that of Thm. 4.1, and the expression analogous to (4.15) is

$$\sum_{j=0}^{n-1} \frac{q_j}{h} \varepsilon_{n-j} = - \sum_{k=0}^n \int_0^1 K(t_k + hs) \tilde{R}_n(t_k + hs) ds + \mathcal{O}(h^5).$$

Taking the second central difference, using the fact that when $K \equiv 1$ the scheme coefficients are

$$q_0 = 5h/8, \quad q_1 = 5h/6, \quad q_2 = 25h/24, \quad \text{and} \quad q_j = h \text{ for } j \geq 3 \quad (5.7)$$

gives an expression like (4.18) with $m = 3$ where the bottom row of the matrix M is now $[-1, 6, 0, 10]/15$. The eigenvalues of M are all distinct and its spectral radius is $\rho(M) = 1$, and this again yields a bound ν_n which satisfies a difference scheme like (4.19) whose final term is $C h^5$. This gives an $\mathcal{O}(h^4)$ bound for each $|\varepsilon_k|$ with $t_k \leq T$, and (5.6) follows. \square

5.3 Stability for discontinuous and highly oscillatory kernels

We first consider the discontinuous kernel introduced in Section 4.2.2. The duration $L = (M + r)h$ is fixed independent of h such that integer $M \geq 5$ and $r \in [0, 1)$. In this case the q_j are given by (5.7) for $j = 0 : M - 2$, the coefficients q_j for $j = M - 1 : M + 2$ are polynomial in r and satisfy

$$q_{M-2} = h \geq q_{M-1} \geq q_M \geq q_{M+1} \geq q_{M+2} \geq 0,$$

and $q_j = 0$ for $j \geq M + 3$.

The stability proof follows that of (Davies & Duncan 2013, §3.1.3); forward differencing (3.4) gives

$$15p_n + 5p_{n-1} + 5p_{n-2} - p_{n-3} + 24 \sum_{j=M-1}^{M+3} \frac{(q_j - q_{j-1})}{h} p_{n-j} = 0 \quad \text{for } n \geq 2,$$

where the first stability factors are $p_0 = 1$, $p_1 = -q_1/q_0 = -4/3$ and we set $p_j = 0$ for $j < 0$. Similar manipulations to those in the previous subsection then give

$$|p_n| \leq \frac{11}{15} Z_{n-1} + \frac{8}{5} Z_{n-(M-1)} \quad \text{for } n \geq 2$$

where now $Z_k = \max_{j \leq k} |p_j|$. It then follows from similar arguments to those used in (Davies & Duncan 2013, §3.1.3) that if $n \geq M - 1$ then

$$Z_n \leq \frac{11}{15} Z_{n-1} + \frac{8}{5} Z_{n-(M-1)} \leq \frac{11}{15} Z_n + \frac{8}{5} Z_{n-(M-1)},$$

i.e. $Z_n \leq 6 Z_{n-(M-1)}$, and $|p_n| \leq Z_n = 4/3$ for $n = 1 : M - 2$. In combination these give the stability bound

$$|p_n| \leq Z_n \leq C 6^{n/(M-1)} \leq C 6^{(T+1)/L}$$

for all $n \leq T/h$ when h is sufficiently small, and so the stability coefficients are bounded independently of h as required by Def. 3.1. Note that this bound is very pessimistic, and in practice the factor is about $1.5^{T/L}$.

This scheme is also stable when applied to the oscillatory kernels $J_0(\omega t)$ and $\cos(\omega t)$ examined in Section 4.2.3. The analysis is much simpler to carry out for this scheme, since there is no problem with poles of $1/Q(\xi)$ on or near the unit circle $|\xi| = 1$. We find that

$$|p_n| \leq C e^{\sigma T} \quad \text{where} \quad C = \begin{cases} 2.2, & K(t) = \cos(\omega t) \\ 1.5, & K(t) = J_0(\omega t) \end{cases}$$

for all $\omega h \in [0, 20\pi]$ and beyond. This is verified in tests computing p_n directly from (3.4) for a finite number of steps $n \leq 2500$ and the same range of values of ωh . They indicate that $|p_n| \leq 1.82$ (for $\cos(\omega t)$) and $|p_n| \leq 4/3$ (for $J_0(\omega t)$), consistent with the estimate above.

5.4 Numerical results

Numerical comparisons of the new modified cubic B-spline approximation for (1.2) are compared with the CQ BDF2 method and with the convolution B-splines with $m = 0, 3$ from Sec. 4 in Figure 5 when a satisfies (2.1) with $d = 4$. The convergence rates for a smooth kernel (left subplot) are as expected: CQ BDF2 has rate $\mathcal{O}(h^2)$; B-splines 0 and 3 have rates $\mathcal{O}(h)$ and $\mathcal{O}(h^2)$; and the modified B-spline 3 has rate $\mathcal{O}(h^4)$. The discontinuous kernel problem (right subplot) has the same convergence rates, apart from the isogeometric $m = 3$ approximation which was shown to be unstable in Section 4.2.2, and is not illustrated.

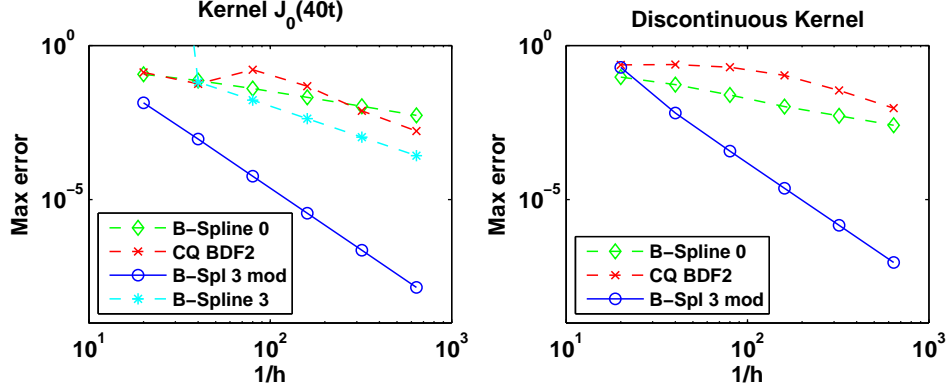


Figure 5: Convergence results for the approximation of (1.2) with maximum time $T = 10$ and $a(t) = t^6 \exp(-50(t - 1/2)^2)$ (see text for details). The stability and $\mathcal{O}(h^4)$ convergence rate of modified B-spline 3 is clear for both problems.

6 Convolution splines for TDBIEs

The results shown here are obtained by approximating the solution of the TDBIE (1.1) in space by piecewise constant basis functions on a generally irregular triangular grid, and in time by convolution splines as

$$u(\mathbf{x}, t_n - t') \approx \sum_{j=0}^n \sum_{k=1}^M u_k^{n-j} \phi_j(t'/h) \eta_k(\mathbf{x}),$$

where the ϕ_j are the modified cubic B-splines from Sec. 5.1,

$$\eta_k(\mathbf{x}) = \begin{cases} 1 & \text{for } \mathbf{x} \in \Gamma_k \\ 0 & \text{otherwise} \end{cases}$$

and Γ_k is the k th triangle on the surface Γ . The spatial Galerkin formulation of the problem gives the time-marching scheme (1.3) with

$$\mathbf{Q}_{j,k}^m = \iint_{\Gamma_j \times \Gamma_k} \frac{\phi_m(|\mathbf{x} - \mathbf{y}|/h)}{|\mathbf{x} - \mathbf{y}|} d\mathbf{x} d\mathbf{y}, \quad \text{and} \quad a_j^n = \int_{\Gamma_j} a(\mathbf{x}, t^n) d\mathbf{x}.$$

The matrices \mathbf{Q}^m are symmetric and calculating each component involves a four dimensional integral. The off-diagonal components and the components a_j^n have smooth integrands and are approximated by a composite triangular quadrature with 16 sub-triangles, each of which is sixth order with 12 quadrature points. Each diagonal element of the \mathbf{Q}^m has a singular integrand, and is first converted into smooth subintegrals using a Duffy-type transformation, and then approximated by the same quadrature rule as the rest of the calculation. Similar results are obtained when piecewise linear spatial basis functions are used.

Figure 6 shows results for (1.1) when Γ is a flat plate and a sphere, both with incident field $a(\mathbf{x}, t) = a_0(t + t_0 - |\mathbf{x}|)/|\mathbf{x}|$, where $a_0(t) = t^4 \exp(-20(t - 1/2)^2)$ for $t > 0$. In these two tests the mesh ratio is chosen to be $h/\Delta x = 1/2$, where Δx is the size of a typical space mesh element. The left-hand graph shows the size (and hence stability) of the approximate solution on a unit square plate. The growth in the L_∞ norm is due to a corner singularity (Holm, Maischak & Stephan 1996) in the exact solution while the maximum of the 1-norm is well-behaved as the mesh is refined. The right-hand plot shows the maximum relative error (i.e. the error normalised by the maximum size of the solution) for scattering from a unit sphere – an exact solution for this problem is given in Sauter & Veit (2011b). The dotted lines show a second order convergence slope, and this is the best which can be expected from the spatial approximation, despite the higher order accuracy of the temporal approximation.

Figure 7 demonstrates the impact of changing the mesh ratio $h/\Delta x$ in the sphere scattering example described above. The left plot shows the maximum relative error in the solution against the number of space elements, and the right plot shows the dependence of this error on h . As one would expect for a scheme with higher order accuracy in time than space, the time error decreases faster than the space error when the mesh is refined with fixed mesh ratio, and the space error eventually dominates. This is clear on the left plot for

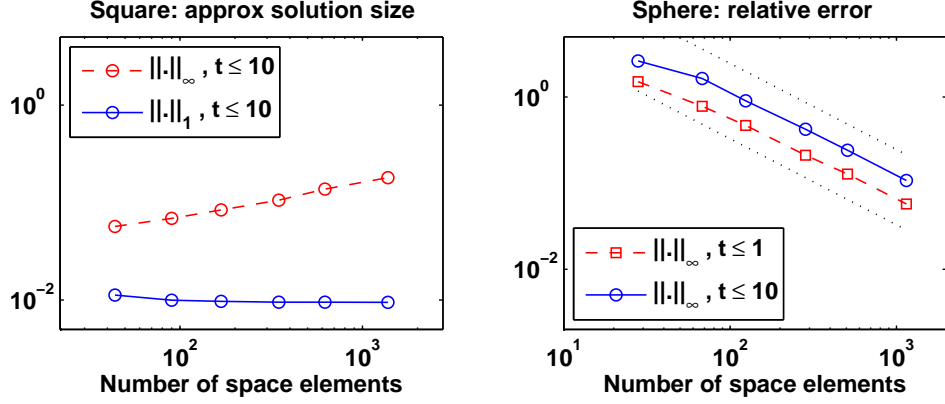


Figure 6: Solution plots for the TBIE (1.1) when the scattering surface Γ is a unit square plate (left plot) and a unit sphere (right plot). See text for details of the space and time approximations used.

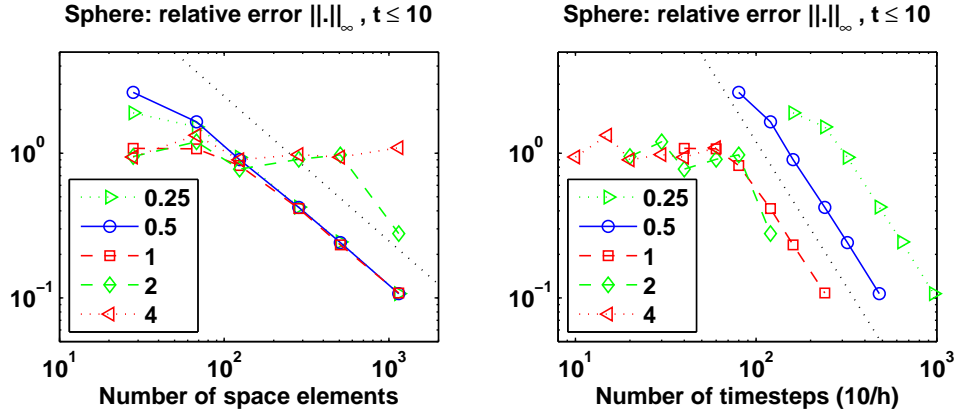


Figure 7: Solution plots for the TBIE (1.1) for mesh ratios $h/\Delta x$ ranging from 0.25 to 4 when the scattering surface Γ is a unit sphere. The dotted line indicates second order convergence. See text for details of the space and time approximations used.

ratios $h/\Delta x = 0.25, 0.5$ and 1 . For the larger mesh ratios the time step size is simply too big to resolve the input function accurately, and the asymptotic convergence regime has not yet been reached. Increasing the mesh ratio decreases the number of time steps (and hence matrices Q^m) used, but increases the number of non-zero entries in each matrix. However the net result is a decrease in the computational cost for both the time marching calculation and the computation of the matrices Q^m .

7 Conclusions

We have derived a new framework for time-stepping approximations of the TDBIE (1.1). The system matrices Q^m of the resulting scheme (1.3) have the same degree of sparsity as for a space-time Galerkin approximation, but are much more straightforward to calculate, especially when higher order (smoother) B-splines are used. The method is constructed as an approximation scheme for the VIE (1.2), and key properties are its backward time aspect (2.9), that the basis functions have compact support and are (mainly) translates, and they satisfy the sum to unity condition (2.12).

These properties permit a full stability analysis for VIEs with kernels which capture some of the important properties of TDBIE problems, as illustrated for B-spline basis functions in Secs. 4.2 and 5.3. In particular the analysis gives a stability bound for the p_n coefficients for the Bessel function kernel (1.4) when $\omega \approx 1/h$. This means that the convergence proof in Davies & Duncan (2003) for the TDBIE (1.1) on $\Gamma = \mathbb{R}^2$ could be applied to an approximation which is a Fourier interpolant in space and a convolution spline in time, without the need to impose an additional (essentially uncheckable) stability assumption. The modified cubic convolution spline

approximation of Sec. 5 is fourth order accurate and gives a very stable approximation for discontinuous and highly oscillatory VIE kernels. The TDBIE numerical test results indicate that the scheme is very stable and performs well.

There is current interest in TDBIE time-stepping methods which can use variable time-steps. See e.g. (López-Fernández & Sauter 2011, López-Fernández & Sauter 2012) for convolution quadrature methods and (Gläflke 2013) for space and time adaptation in the full space-time Galerkin method for scattering problems in 2D space. Because our TDBIE time-stepping method is based on B-splines (whose key approximation properties are retained for a non-uniform knot distribution) and standard piecewise polynomials in space, the various strategies described in (Gläflke 2013) for space and time adaptation could be applied. However, using variable time-stepping in any TDBIE approximation algorithm imposes an overhead because it essentially involves recalculating the Q^m matrices of (1.3) at every time step, and this is extremely expensive.

We note that there are also other choices of basis functions which seem to give stable approximations of (1.2) and (1.1) when used in the same convolution framework. Non-polynomial temporal basis functions ϕ_j are introduced in Davies & Duncan (2013); they are translates for $j \geq 2$, and for $j = 0 : 1$ they are defined as described in Wu & Schaback (1994) for radial basis function (RBF) multi-quadrics in order to ensure that quasi-interpolation in terms of $\{\phi_j\}_{j=0}^\infty$ is linearity-preserving. They also work well as a temporal approximation of the TDBIE (1.1), but because the basis functions are global the system matrices need to be sparsified (but this is straightforward because they are highly peaked). The method derived in (Davies & Duncan 2013) is second order accurate, and the fourth order modified cubic B-spline approximation of Sec. 5 is a significant improvement. Extension of this approach to modified B-splines with $m > 3$ is work in progress.

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